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- There is a classification for unimodular lattices by means of the rank, signature, the parity of Q, and (in)definiteness, etc.
  For indefinite lattices the classification is complete: $m\mathbf{1} \oplus n(-\mathbf{1})$ (Odd) or $mH \oplus nE_8$ (Even).
  For definite lattices it is more involved; finitely many types in each rank but only known up to some ranks.
  e.g. The case $d = 4$ above is an example of a $K3$ surface (compact, simply connected complex surface with zero 1st Chern class). In this case $Q = 3H \oplus 2(-E_8)$.

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- **(Donaldson)** For any simply connected complex surface S with $b^+(S) > 3$ there exists an oriented smooth closed 4-manifold homotopy equivalent to S but not diffeomorphic to any complex surface.
- (Quote from Mareño's notes) The correlation function of the observables of twisted $\mathcal{N} = 2$ Yang-Mills theory is precisely the corresponding Donaldson invariant.

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- If $P \to X$ is a principal G-bundle and $\rho\colon G \to \mathrm{GL}(V)$ is a representation one can associate a vector bundle $E := P \times_G V \to X$ with fiber V. For example, the adjoint bundle $\operatorname{ad} P \to X$ is the bundle associated to the adjoint representation $\rho\colon G \to \mathrm{GL}(\mathfrak{g})$.

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- Three equivalent ways of thinking of a connection on P:
  1) A 1-form A on P with values in $\mathfrak{g}$, i.e. $A \in \Omega^1(P, \mathfrak{g})$, which is invariant under the action of G on P and the adjoint action of G on $\mathfrak{g}$, and restricts to the canonical right invariant from on each fiber of P.
  2) A choice of a field of G-invariant horizontal subspaces $H_A \subset T_P$ that are transversal to the fibers of P: $T_P = H_A \oplus T_{P/X}$.
  3) As a covariant derivative $\nabla_A\colon \Omega^0(X, E) \to \Omega^1(X, E)$.
  (satisfying the Leibniz rule: $\nabla_A(f\sigma) = df \otimes \sigma + f\nabla_A(\sigma)$ for any section $\sigma \in \Omega^0(X, E)$ and function $f \in \Omega^0(X)$.)

- The difference of two connections $\nabla_A - \nabla_{A'}$ is a tensor i.e. an element of $\Omega^1(X, \operatorname{ad} P)$ by viewing the adjoint bundle ad P as a subbundle of End E. Conversely, $\nabla_A + a$ is again a connection for any $a \in \Omega^1(X, \operatorname{ad} P)$, where $\Omega^1(X, \operatorname{ad} P)$ acts via the contraction

$$\Omega^0(X, E) \times \Omega^1(X, \operatorname{End} E) \to \Omega^1(X, E).$$

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- Curvature of a connection: $F_A := \nabla_A \circ \nabla_A \in \Omega^2(X, \operatorname{ad} P)$.
- Gauge group: $\mathcal{G} := \operatorname{Aut} E$. It is an infinite dimensional Lie group with the Lie algebra $\Omega^0(X, \operatorname{ad} P)$.
  $\mathcal{G}$ acts on $\mathcal{A}$ by the rule

$$\forall u \in \mathcal{G}, \sigma \in \Omega^0(X, E) \quad \nabla_{u(A)}\sigma = u\nabla_A(u^{-1}\sigma).$$

- Now suppose X is equipped with a Riemannian metric g and a volume form ω. This gives a Hodge operator $\star\colon H^2(X,\mathbb{R}) \to H^2(X,\mathbb{R})$ characterized by

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- The splitting $\Omega^2(X)$ into $\Omega^{2,\pm}(X)$ naturally extends to the splitting of $\Omega^2(X, \mathrm{ad}\, P)$ into $\Omega^{2,\pm}(X, \mathrm{ad}\, P)$. So $F_A = F_A^+ + F_A^-$.
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  ASD connection: A connection A is called ASD if $F_A^+ = 0$.
- Importance: When $G = \mathrm{SU}(2)$ and $c_2(E) > 0$ ASD connections minimize the Yangs-Mills functional $S_{YM} = \int_X |F_A|^2 = \int_X |F_A^-|^2 + \int_X |F_A^+|^2$.
  This is because on the Lie algebra of skew adjoint trace free matrices $\mathrm{Tr}(\xi^2) = -|\xi|^2$, and so $8\pi^2 c_2(E) = \int_X \mathrm{Tr}(F_A^2) = \int_X |F_A^-|^2 - \int_X |F_A^+|^2$ is a lower bound of S_{YM} and it is achieved if and only if A is ASD.

- Let E be either a complex rank 2 SU(2)-bundle or a real rank 3 SO(3)-bundle. In the former case E is classified by its second Chern class $c_2(E)$ and in the latter case by its first Pontriagin class $p_1(E)$ and the second Stiefel-Whitney class $w_2(E)$. We will concentrate on the former case.
  Let $c_1 := c_1(E)$ and $c_2 := c_2(E) \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$.

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- Let $\mathcal{A}^* \subset \mathcal{A}$ be the subspace of irreducible connections, i.e. there are no decompositions $E = L_1 \oplus L_2$ and $\nabla_A = \nabla_{A_1} \oplus \nabla_{A_2}$; let

$$N_g(c_1, c_2) := \{A \in \mathcal{A}^* \mid A \text{ is ASD}\}/\mathcal{G}.$$

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- **Theorem:** If g is generic $N_g(c_1, c_2)$ is a smooth oriented manifold of dimension $d = 8c_2 - 2c_1 - 3(1 - b_1(X) + b^+(X))$.

- Let X be a projective algebraic surface with an ample divisor H, Fubini-Study metric g, and Kähler form ω. Then

$$\Lambda^+(X) = \mathrm{Re}\left(\Lambda^{2,0}(X) \oplus \Lambda^{0,2}(X)\right) \oplus \mathbb{R}\omega, \qquad \Lambda^-(X) = \omega^\perp \subset \Lambda^{1,1}(X).$$

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- In this case $\nabla_A = \partial_A + \overline{\partial}_A$ according to $\Omega^1(X,E) = \Omega^{1,0}(X,E) \oplus \Omega^{0,1}(X,E)$ and
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- **(Donaldson)** (1) defines a holomorphic structure on E, and (2) says that E is μ-stable with respect to H i.e.
for any sub-line-bundle $F \subset E \quad H \cdot c_1(F) < \frac{H \cdot c_1}{2}$.

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for any sub-line-bundle $F \subset E \quad H \cdot c_1(F) < \frac{H \cdot c_1}{2}$.
- Let $M_H^L(c_2)$ be the moduli space of rank 2 μ-stable holomorphic bundles with fixed determinant L such that $c_1(L) = c_1$ and fixed second Chern class c_2. The (expected) real dimension of $M_H^L(c_2)$ is

$$2(\chi^h(\mathcal{O}_X) - \chi^h(E)) = 8c_2 - 2c_1^2 - 6(1 - h^{0,1}(X) + h^{0,2}(X)) = d,$$

by noting that $b_1(X) = 2h^{0,1}(X)$ and $b^+(X) = 2h^{0,2}(X) + 1$.

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  by noting that $b_1(X) = 2h^{0,1}(X)$ and $b^+(X) = 2h^{0,2}(X) + 1$.
- **(Donaldson)** There exists a homeomorphism $\Phi\colon N_g(c_1, c_2) \to M_H^L(c_2)$.

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- **(Li, Morgan)** There exists a morphism $\overline{\Phi}\colon \overline{M_H^L(c_2)} \to \overline{N_g}(c_1, c_2)$ extending Donaldson's homeomorphism Φ. Moreover, $\overline{\Phi}_*[\overline{M_H^L}(c_2)] = [\overline{N_g}(c_1, c_2)]$.

- For simplicity, assume X is simply connected and also there are a universal bundle $\mathcal{E}$ over $X \times \overline{N_g}(c_1, c_2)$ and a universal connection $\mathcal{D}\colon \Omega^0(\mathcal{E}) \to \Omega^1(\mathcal{E})$. Define

$$\mu\colon H_*(X) \to H^*(\overline{N_g}(c_1, c_2)) \qquad \mu(\alpha) := \frac{1}{4}\big(c_2(\mathcal{E}) - c_1^2(\mathcal{E})\big)/\alpha.$$

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- Let $\alpha_1, \ldots, \alpha_l \in H_2(X)$ and $p \in H_0(X)$ be the class of a point. Define the *Donaldson invariant* by

$$\big\langle \alpha_1, \ldots, \alpha_l, p^m \big\rangle_d^{c_1, g} := \int_{[\overline{N_g}(c_1, c_2)]} \mu(\alpha_1) \cup \cdots \cup \mu(\alpha_l) \cup \mu(p)^m.$$

This is nonzero only if $2l + 4m = d$ (dimension of $\overline{N_g}(c_1, c_2)$).

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  This is nonzero only if $2l + 4m = d$ (dimension of $\overline{N_g}(c_1, c_2)$).
- If $b^+(X) > 1$, Donaldson invariants are independent of the choice of the generic metric, and so they are really the invariants of the differentiable structure of X.

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  This is nonzero only if $2l + 4m = d$ (dimension of $\overline{N_g}(c_1, c_2)$).
- If $b^+(X) > 1$, Donaldson invariants are independent of the choice of the generic metric, and so they are really the invariants of the differentiable structure of X.
- If $b^+(X) = 1$, the invariants depend only on a system of walls and chambers in $H^2(X, \mathbb{R})^+ := \{\alpha \in H^2(X, \mathbb{R}) \mid \alpha^2 > 0\}$.
  Göttsche-Yoshioka-Nakajima proved wall-crossing formulas involving modular forms.

- For simplicity, assume X is simply connected and also there are a universal bundle $\mathcal{E}$ over $X \times \overline{N_g}(c_1, c_2)$ and a universal connection $\mathcal{D}\colon \Omega^0(\mathcal{E}) \to \Omega^1(\mathcal{E})$. Define

$$\mu\colon H_*(X) \to H^*(\overline{N_g}(c_1, c_2)) \qquad \mu(\alpha) := \frac{1}{4}\big(c_2(\mathcal{E}) - c_1^2(\mathcal{E})\big)/\alpha.$$

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  Göttsche-Yoshioka-Nakajima proved wall-crossing formulas involving modular forms.
- If X is a projective algebraic surface as before, one can define Donaldson invariants algebraically by replacing $\mathcal{E}$ with the universal sheaf over $X \times \overline{M_H^L}(c_2)$. Using the map $\overline{\Phi}$ above one can see that the two types of invariants coincide.

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- Let $S_*(X) = \mathrm{Sym}(H_2(X) \oplus H_0(X))$. It is graded by assigning degree 2 (resp. degree 4) to the elements of $H_2(X)$ (resp.$H_0(X)$). Then Donaldson invariants define a map $\langle - \rangle_d^{c_1} \colon S_d(X) \to \mathbb{Q}$. One can then define

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- For $a \in H_2(X)$ and $\alpha \in S_*(X)$ and a variable z write

$$D_{X,c_1}(\alpha e^{az}) := \sum_{n \geqslant 0} D_{X,c_1}(\alpha a^n) z^n/n!.$$

- **(Kronheimer-Mrowka)** If X is of simple type there exit so called *basic classes* $K_1, \ldots, K_l \in H^2(X, \mathbb{Z})$ and rational numbers $q_1(c_1), \ldots, q_l(c_1) \in \mathbb{Q}$ such that for all $a \in H_2(X)$

$$D_{X,c_1}((1+p/2)e^{az}) = e^{a^2z^2/2} \sum_{i=1}^{l} q_i(c_1) e^{\langle K_i, a \rangle z}.$$

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- (Example $X = K3$) The only basic class is $K_1 = 0$ and for all $a \in H_2(X, \mathbb{Z})$

$$D_{K3,c_1}((1 + p/2)e^{az}) = \frac{(-1)^{c_1^2/2}}{2} e^{a^2 z^2/2}.$$

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- (Mariño notes) The correlation functions of $\mathcal{N} = 2$ Yang-Mills theory coupled to hypermultiplets coincides with SW invariants.
- We saw that Donaldson invariants of X are independent of metric when $b^+(X) > 1$ and depend mildly on metric when $b^+(X) = 1$. The same is true for SW invariants as we will see. This is a feature of the Witten type TFT's (compared to Scwharz type) that there is an explicit metric dependence in defining the theory but the correlation functions happen not to depend (or depend mildly) on the metric.

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  $\exists$ lift of the structure to the double cover $\mathrm{Spin}(n) \to \mathrm{SO}(n) \Leftrightarrow w_2(P) = 0$.
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- If $\{e_1, e_2, e_3, e_4\}$ is the standard basis of $\mathbb{R}^4$ then $\mathrm{Cl}(\mathbb{R}^4)$ is an $\mathbb{R}$-algebra generated by e_i's subject to the relations $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ for $i \neq j$. Its vector space dimension is 16 ($\mathbb{R}$-basis: $\{e_{i_1} \cdots e_{i_t}\}_{i_1 < \cdots < i_t}$). The parity of t defines a $\mathbb{Z}_2$-grading $\mathrm{Cl}(\mathbb{R}^4) = \mathrm{Cl}_0(\mathbb{R}^4) \oplus \mathrm{Cl}_1(\mathbb{R}^4)$. There is an identification of algebras $\mathrm{Cl}_0(\mathbb{R}^4) \cong \mathbb{H} \oplus \mathbb{H}$. The group $\mathrm{Spin}(4)$ is identified with the subgroup of $\mathrm{Cl}_0^\times(\mathbb{R}^4)$ generated by $v \in \mathbb{R}^4$ with $|v| = 1$.

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- There is a natural linear isomorphism

$$\Lambda^*\mathbb{R}^4 \to \mathrm{Cl}(\mathbb{R}^4) \quad e_{i_1} \wedge \cdots \wedge e_{i_t} \mapsto e_{i_1} \cdots e_{i_t}.$$

  Let $w := -e_1 e_2 e_3 e_4$; it satisfies $w^2 = 1$. Let $(\mathrm{Cl}(\mathbb{R}^4) \otimes \mathbb{C})^\pm$ be the ± 1-eigenspaces of the left multiplication by w on $\mathrm{Cl}(\mathbb{R}^4) \otimes \mathbb{C}$. Under the isomorphism above $(\mathrm{Cl}_0(\mathbb{R}^4) \otimes \mathbb{C})^+$ corresponds to

$$\mathbb{C}(\frac{1+w}{2}) \oplus (\Lambda^{2,+}\mathbb{R}^4) \otimes \mathbb{C}.$$

- If X is a spinable manifold let $\widetilde{P} \to X$ be a corresponding double cover of the frame bundle of T_X. There exists an associated complex spinor bundle

$$S := \widetilde{P} \times_{\mathsf{Spin}(4)} \Delta_{\mathbb{C}}(\mathbb{R}^4),$$

where $\Delta_{\mathbb{C}}(\mathbb{R}^4)$ is the unique (up to isomorphism) complex representation of the Clifford algebra $\mathrm{Cl}(\mathbb{R}^4)$ (using the identification $\mathrm{Cl}(\mathbb{R}^4) \otimes \mathbb{C} \cong M_2(\mathbb{C})$).

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In the $+$ case the identity endomorphism corresponds to $\frac{1+w}{2}$.
- The decomposition of $\Delta_{\mathbb{C}}(\mathbb{R}^4) = \Delta_{\mathbb{C}}^{+}(\mathbb{R}^4) \oplus \Delta_{\mathbb{C}}^{-}(\mathbb{R}^4)$ induces the decomposition of the spinor bundle

$$S = S^+ \oplus S^-$$

into the so called $\pm$-*chirality spinors*. The $\mathrm{SU}(2)$-bundles $S^{\pm}$ can be alternatively obtained via the identification $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ (induced by $\mathrm{Cl}_0(\mathbb{R}^4) \cong \mathbb{H} \oplus \mathbb{H}$).

- It turns out there are no obstructions for constructing a spin^c structure on X, i.e. a lift of the structure of T_X to the double cover $\mathrm{Spin}^c(4) \to \mathrm{SO}(4) \times \mathrm{U}(1)$. $\mathrm{Spin}^c(4)$ is the subgroup of $(\mathrm{Cl}_0(\mathbb{R}^4) \otimes \mathbb{C})^\times$ generated by $\mathrm{Spin}(4)$ and the unit circle in $\mathbb{C}$.

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- The projection $\mathrm{Spin}^c(4) \to \mathrm{U}(1)$ determines a complex line bundle $\mathcal{L}$, such that $c_1(\mathcal{L}) \equiv_2 w_2(X)$. It is called the *determinant* of the spin^c-structure. One can similarly define the spinor bundle

$$S_{\mathcal{L}} := \widetilde{P} \times_{\mathrm{Spin}^c(4)} \Delta_{\mathbb{C}}(\mathbb{R}^4),$$

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- There is a bijection between the spin^c structures on X and the elements of $H^2(X, \mathbb{Z})$. Varying a given spin^c structure by a class $\alpha \in H^2(X, \mathbb{Z})$ amounts to replacing $S_{\mathcal{L}}$ by $S_{\mathcal{L} \otimes L_{2\alpha}} \cong S_{\mathcal{L}} \otimes L_\alpha$, where L_α is the complex line bundle with $c_1(L_\alpha) = \alpha$.

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- When X has an almost complex structure compatible with the Riemannian metric then there is a canonical choice of spin^c structure given by $\mathcal{L} = K_X^{-1}$, the inverse of the canonical bundle of $(0,2)$-forms. In this case, $S^+_{K_X^{-1}} \cong \oplus_{i=0}^1 T_X^{0,2i}$ and $S^-_{K_X^{-1}} \cong T_X^{0,1}$.

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- ω and A determine a connection on the principal $\mathrm{SO}(4) \times \mathrm{U}(1)$-bundle $\widetilde{P}/\{\pm 1\}$, and there is unique lift of this connection to its double cover $\widetilde{P}$.
- Let $\widetilde{\nabla} \colon \Omega^0(X, S_{\mathcal{L}}) \to \Omega^1(X, S_{\mathcal{L}})$ be the induced covariant derivative on the spinor bundle.

- Let $\mathrm{Cl}(X) := P \times_{\mathrm{SO}(4)} \mathrm{Cl}(\mathbb{R}^4)$ be the associated bundle of Clifford algebras. Since X is Riemannian there is a canonical identification $T_X \cong T_X^*$. Thus, $\mathrm{Cl}(X)$ can be viewed as a new algebra structure on $\Lambda^* T_X^*$ in addition to its own exterior algebra structure.

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- Define the Dirac operator

$$\partial_A \colon \Omega^0(X, S_{\mathcal{L}}) \to \Omega^0(X, S_{\mathcal{L}}) \qquad \partial_A(\sigma)(x) = \sum_{i=1}^{n} e_i \cdot \widetilde{\nabla}_{e_i}(\sigma)(x),$$

where $\{e_1, \ldots, e_n\}$ is an oriented orthonormal frame for $T_{X,x}$ and $\cdot$ is the Clifford multiplication. This definition is independent of the choice of $\{e_1, \ldots, e_n\}$.

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where $\{e_1, \ldots, e_n\}$ is an oriented orthonormal frame for $T_{X,x}$ and $\cdot$ is the Clifford multiplication. This definition is independent of the choice of $\{e_1, \ldots, e_n\}$.
- If X is a Kähler manifold, there is a unique hermitian connection A on K_X^{-1} and Dirac operator simplifies to

$$\partial_A \colon \oplus_{i=0}^{1} \Omega^{0,2i}(X, \mathbb{C}) \to \Omega^{0,1}(X, \mathbb{C})$$
$$\partial_A(\sigma)(x) = \sqrt{2}\big(\overline{\partial}(\sigma)(x) + \overline{\partial}^*(\sigma)(x)\big).$$

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- The SW equations are for a spinor field $\psi \in \Omega^0(X, S^+_{\mathcal{L}})$ and a U(1)-connection A on $\mathcal{L}$:

$$\begin{cases} F_A^+ = \psi \otimes \psi^* - \frac{|\psi|^2}{2}\,\mathrm{Id}, \\ \partial_A(\psi) = 0. \end{cases}$$

Here, $\psi \otimes \psi^*$ is a section of

$$S^+_{\mathcal{L}} \otimes (S^+_{\mathcal{L}})^* \cong \mathrm{End}\, S^+_{\mathcal{L}} \cong (\mathrm{Cl}_0(P) \otimes \mathbb{C})^+ \cong \mathbb{C}(\frac{1+w}{2}) \oplus \Lambda^{2,+}(T_X) \otimes \mathbb{C},$$

where $\frac{1+w}{2}$ acts as the identity. The right hand side of the first equation is traceless and hence is identified with a section of $\Lambda^{2,+}(T_X) \otimes \mathbb{C}$. Finally, identifying $T_X \cong T_X^*$ by using the metric, one can think of the right hand side as a self-dual 2-form.

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- (Theorem) $\mathcal{B}(\widetilde{P})$ is a Hilbert manifold, and it is homotopy equivalent to $\mathbb{CP}^\infty \times K(H^1(X, \mathbb{Z}), 1)$. There is a universal S^1-bundle over $\mathcal{B}(\widetilde{P})$ corresponding to the $\mathbb{CP}^\infty$-factor. Let $\mu \in H^2(\mathcal{B}(\widetilde{P}), \mathbb{Z})$ be its first Chern class.

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- (Main theorem) Suppose $b^+(X) > 0$. For a generic h, the perturbed moduli space $\mathcal{M}(\mathcal{P}, h)$ forms a smooth compact submanifold of $\mathcal{B}(\widetilde{P})$ of dimension

$$\frac{c_1(\mathcal{L})^2 - 2\chi(X) - 3\sigma(X)}{4},$$

  where $\chi(X)$ is the Euler characteristic, and $\sigma(X) = b^+(X) - b^-(X)$ is the signature of X.

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- For a generic h these orientations provides $\mathcal{M}(\mathcal{P}, h)$ with an orientation. If $d = \dim \mathcal{M}(\mathcal{P}, h)$, define

$$\mathrm{SW}(\widetilde{P}) := \begin{cases} \int_{\mathcal{M}(\mathcal{P},h)} \mu^{d/2} & d \in 2\mathbb{Z}, \\ 0 & d \notin 2\mathbb{Z}. \end{cases}$$

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- Let $\mathrm{Spin}^c(X)$ be the set of isomorphism classes of spin^c structures $\widetilde{P} \to X$. We get SW invariants

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- (Involution) $\widetilde{P}$ is a double cover of $P_{\mathrm{SO}(n)} \times_X P_{S^1}$. Let $P^*_{S^1}$ be the dual (conjugate) bundle of P_{S^1}. The pullback of $\widetilde{P}$ via

$$(\mathrm{Id}, \iota)\colon\ P_{\mathrm{SO}(n)} \times_X P^*_{S^1} \to P_{\mathrm{SO}(n)} \times_X P_{S^1}$$

  induces a spin^c-structure denoted by $-\widetilde{P}$. There is a natural homeomorphism $\mathcal{M}(\widetilde{P}, h) \to \mathcal{M}(-\widetilde{P}, -h)$ and moreover,

$$\mathrm{SW}(-\widetilde{P}) = (-1)^{\frac{1+b^+(X)-b_1(X)}{2}} \mathrm{SW}(\widetilde{P}).$$

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- Suppose that $H^1(X, \mathbb{Z}) = 0$ and $c_1(\mathcal{L}) \neq 0$. For any metric g there is a unique g-self dual harmonic 2-form $\omega^+(g)$ lying in the positive component of $H^2(X, \mathbb{R})^+$. Let $\mathcal{R}_\pm$ be the space of Riemannian metrics g on X such that $\omega^+(g) \cdot c_1(\mathcal{L}) > 0$ or < 0.

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- $\mathrm{SW}_g(\widetilde{P})$ is constant on $\mathcal{R}_\pm$, so we can simply write $\mathrm{SW}_\pm(\widetilde{P})$. Moreover, if $d \in 2\mathbb{Z}$
$$\mathrm{SW}_+(\widetilde{P}) - \mathrm{SW}_-(\widetilde{P}) = (-1)^{1+d/2}.$$

- Suppose (X, ω) is a Kähler surface with Kähler metric. Let $\widetilde{P}_J$ be the spin^c structure with the determinant K_X^{-1}. As we have seen,

$$S^+_{K_X^{-1}} \cong \oplus_{i=0}^1 T_X^{0,2i}, \qquad S^-_{K_X^{-1}} \cong T_X^{0,1}$$

and $\sqrt{2}(\overline{\partial} + \overline{\partial}^*) \colon \Omega^0(X, \mathbb{C}) \oplus \Omega^{0,2}(X, \mathbb{C}) \to \Omega^{0,1}(X, \mathbb{C})$.

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- Any other spin^c structure $\widetilde{P}$ differs from this by tensoring with a $\mathrm{U}(1)$-bundle $\mathcal{L}_0$, with the new determinant bundle $\mathcal{L} = K_X^{-1} \otimes \mathcal{L}_0^2$ and the spinors bundles

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- Let $\psi = (\alpha, \beta) \in \Omega^0(X, \mathcal{L}_0) \oplus \Omega^{0,2}(X, \mathcal{L}_0)$ SW equation can be written as

$$\begin{cases} (F_A^+)^{1,1} = \frac{i}{4}(|\alpha|^2 - |\beta|^2)\omega, \\ F_A^{0,2} = \frac{\overline{\alpha}\beta}{2}, \\ \sqrt{2}(\overline{\partial_{A_0}}(\alpha) + \overline{\partial_{A_0}}^*(\beta)) = 0. \end{cases}$$

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- For any solution (A, ψ), A induces a holomorphic structure on $\mathcal{L}$ and hence on $\mathcal{L}_0$, with respect to which α is a holomorphic section of $\mathcal{L}_0$ and $\overline{\beta}$ is a holomorphic section of $K_X \otimes \mathcal{L}_0^{-1}$. If $\deg \mathcal{L} = \int_X c_1(\mathcal{L}) \wedge \omega \leqslant 0$ (resp. $\geqslant 0$) $\beta = 0$ (resp. $\alpha = 0$). In particular, if $\deg \mathcal{L} = 0$ then any solution consists of an ASD connection A on $\mathcal{L}$.

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- If X is minimal algebraic surface of general type, then for any Kähler metric

$$\mathrm{SW}(\widetilde{P}) = \begin{cases} 1 & \widetilde{P} = \widetilde{P}_{K_X^{-1}}, \\ (-1)^{1-h^{0,1}(X)+h^{0,2}(X)} & \widetilde{P} = -\widetilde{P}_{K_X^{-1}}, \\ 0 & \text{otherwise.} \end{cases}$$

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- If X is minimal Kähler surface which is elliptic and K_X is not a torsion class. Then

$$\mathrm{SW}(\widetilde{P}_{K_X^{-1}}) = 1, \qquad \mathrm{SW}(-\widetilde{P}_{K_X^{-1}}) = (-1)^{1-h^{0,1}(X)+h^{0,2}(X)}.$$

  Furthermore, if $\mathrm{SW}(\widetilde{P}) \neq 0$ then the image of $c_1(\mathcal{L})$ in $H^2(X, \mathbb{Q})$ is a rational multiple between -1 and 1 of the image of K_X.

- Suppose that X is a projective surface and $\beta \in H^2(X, \mathbb{Z})$. Define $H_\beta(X)$ to be the moduli space of pairs (L, s) of a nonzero holomorphic line bundle and a holomorphic section such that $c_1(L) = \beta$. Equivalently, $H_\beta(X)$ is Grothendieck's Hilbert scheme of divisors $D \subset X$ in class β, and so is a projective scheme.

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- **(Dürr-Kabanov-Okonek)** There is a virtual fundamental class

$$[H_\beta(X)]^{\text{vir}} \in A_{\text{vd}}(H_\beta(X)), \quad \text{vd} := \beta(\beta - K_X)/2.$$

Note that for the spin^c structure $\widetilde{P}$ with $c_1(\mathcal{L}) = 2\beta - K_X$ we have $\dim \mathcal{M}(\widetilde{P}, h) = \frac{c_1(\mathcal{L})^2 - 2\chi(X) - 3\sigma(X)}{4} = 2\,\text{vd}$ using the identity $K_X^2 = 2\chi(X) + 3\sigma(X)$.

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- For $\beta \in H^2(X, \mathbb{Z})$ let $\beta^\vee := K_X - \beta$ (this plays the role of involution in SW theory). Let

$$\rho\colon H_\beta(X) \to \text{Pic}_\beta(X), \qquad \rho^\vee : H_{\beta^\vee}(X) \to \text{Pic}_\beta(X)$$

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- Let $\mathbb{D} \subset X \times H_\beta(X)$ and $\mathbb{D}^\vee \subset X \times H_{\beta^\vee}(X)$ be the universal divisors. Define

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- Poincaré invariants: $(I, I^\vee)\colon H^2(X, \mathbb{Z}) \to \Lambda^* H^1(X, \mathbb{Z}) \times \Lambda^* H^1(X, \mathbb{Z})$ defined by
$$I(\beta) := \rho_*\Big(\sum_i \mu^i \cap [H_\beta(X)]^{\mathrm{vir}}\Big),$$
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These are both zero if β is not a type $(1,1)$ class.

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These are both zero if β is not a type $(1,1)$ class.
- Theorem: If $b^+(X) > 1$ then $I(\beta) = I^\vee(\beta) = \text{SW}(\widetilde{P})$.
If $b^+(X) = 1$ then $I(\beta) = \text{SW}_+(\widetilde{P})$ and $I^\vee(\beta) = \text{SW}_-(\widetilde{P})$.
Here, SW invariants are defined with respect to the canonical orientation data and $\widetilde{P}$ is the spin^c structure with determinant $2\beta - K_X$ i.e. defers from the canonical spin^c structure $\widetilde{P}_{K_X^{-1}}$ by a U(1)-bundle in class β.

- Let X be a smooth closed oriented 4-manifold with $b_1(X) = 0$ and $b^+(X) \geqslant 3$ and odd. Let $\beta \in H^2(X, \mathbb{Z})$ and $\alpha \in H_2(X, \mathbb{Q})$ and $p \in H_0(X)$ be the class of a point.

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- **(Witten)** $D_{X,\beta}((1 + p/2)e^{\alpha z}) =$
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where the sum is over all the spin^c structures and $c_1(\mathfrak{s})$ is the first Chern of the determinant line bundle of $\mathfrak{s}$.

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- Another approach uses the moduli space of SO(3)-monopoles (higher rank analog of U(1)-monopoles in SW theory) by Pidstrigach-Tyurin and Feehan-Leness, which in series of papers proves Witten's formula under certain conditions on X, e.g. when $7\chi(X) + 11\sigma(X) \geqslant 12$. The idea is to construct a cobordism between links of the compact moduli spaces of U(1) monopoles of Seiberg-Witten type and the Donaldson moduli space of ASD connections, which appear as singularities in the larger moduli space of SO(3)-monopoles.

- **(Mochizuki)** In case X is a projective complex surface

$$D_{X,\beta}(\alpha^k p^l) = \sum_{\mathfrak{s}} f_{k,l}(\chi^h(X), K_X^2, \mathfrak{s}, \beta, \alpha)\, \mathrm{SW}(\mathfrak{s}),$$

where $f_{k,l}(-)$'s are (non-explicit) universal polynomials. This formula is obtained by the sheaf theoretic approach to Donaldson theory. A master moduli space is constructed equipped with a $\mathbb{C}^*$-action whose fixed locus is a union of moduli space of rank 2 semistable sheaves and products of Seiberg-Witten moduli space and Hilbert schemes of points on X. Mochizuki's formula is then an application of the (virtual) Atiyah-Bott localization formula.

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- **(Göttsche-Yoshioka-Nakajima)** gave an interpretation of $f_{k,l}(-)$'s in terms of the invariants of Nekarsov's framed moduli spaces of torsion free sheaves on $\mathbb{P}^2$, which are "deformed partition function for the N = 2 SUSY gauge theory with a single fundamental matter". These are in turn shown to be given by certain period integrals over Seiberg-Witten curves, which can be written as the residue of some differential forms. A subtle analysis of these around the poles leads to a proof of Witten's formula for projective surfaces. This approach is very similar to their approach to the wall-crossing problem discussed in the next Section.

- Recall Donaldson invariants are defined by the moduli space of ASD connections on a principal SU(2)- or SO(3)-bundle on a closed oriented smooth 4-manifold X. ASD requirement depends on the choice of a Riemannian metric g. For generic g there are no reducible ASD connections and the moduli spaces are smooth manifolds.

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- The *wall-crossing terms* are the differences of Donaldson invariants when the metric moves to another chamber by passing through a wall. **Kotschick-Morgan** conjectured a *polynomiality property* for the wall-crossing terms.
- **Moore-Witten** derived a wall-crossing formula based on Seiberg-Witten ansatz of the $\mathcal{N} = 2$ supersymmetric Yang-Mills theory on $\mathbb{R}^4$. Modular forms appear in their wall-crossing terms due to the family of elliptic curves parameterized by the u-plane. This argument has not be mathematically justified yet. We will discuss another approach introduced by **Nekrasov**.

- Suppose $b^+(X) = 1$. Any $0 \neq \xi \in H^2(X, \mathbb{Z})$ determines a wall $W^\xi := \{x \in \mathcal{C} \mid x \cdot \xi = 0\}$.
  For $c_1 \in H^2(X, \mathbb{Z})$ and $d \in \mathbb{Z}$ if $\xi + c_1$ is divisible by 2 in $H^2(X, \mathbb{Z})$ and also $d + 3 + \xi^2 \geqslant 0$ then W^ξ is called a wall of type (c_1, d). If only the first condition is satisfied W^ξ is called a wall of type c_1.
  The chambers of (c_1, d) are the connected components of the complement of all the walls of type (c_1, d) in $\mathcal{C}$. The Donaldson invariant $D^g_{c_1,d}(\alpha^n, p^b)$ only depends on the chamber of $\omega(g)^+$.

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- If $C_\pm$ are two chambers of type (c_1, d) and $g_\pm$ are Riemannian metrics with $\omega(g_\pm)^+ \in C_\pm$ then

$$D^{g_+}_{c_1,d}(\alpha^n, p^b) - D^{g_-}_{c_1,d}(\alpha^n, p^b) = \sum_\xi \Delta_{\xi,d}(\alpha^n, p^b),$$

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- From now on we assume X is a smooth projective surface and H is an ample divisor on X. The cohomology class of H is represented by $\omega(g)^+$ when g is the Fubini-Study metric associated to H.
  As we have seen, the Donaldson invariant $D^H_{c_1,d}(\alpha^n, p^b)$ can be computed by the moduli space $M_H(c_1, d)$ of rank 2 semistable torsion free sheaves E with $c_1(E) = c_1$ and $4c_2(E) - c_1(E)^2 - 3 = d$.

- Let $\ell \subset \mathbb{P}^2$ be the line at infinity, and for any integer n let $M(n)$ be the moduli space of pairs (E, ϕ), where E is a rank 2 torsion free sheaf on $\mathbb{P}^2$ with $c_2(E) = n$, which is a vector bundle in a neighborhood of ℓ, and $\phi \colon E|_\ell \to \mathcal{O}_\ell^2$ is an isomorphism.

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- The fixed point set $M(n)^{\widetilde{T}}$ is the set of $(E, \phi) = (I_{Z_1}, \phi_1) \oplus (I_{Z_2}, \phi_2)$, where I_{Z_i} are Γ-fixed ideals of points (0-dimensional subschemes) in $\mathbb{C}^2 = \mathbb{P}^2 \backslash \ell$ such that $\mathrm{len}(Z_1) + \mathrm{len}(Z_2) = n$, and ϕ_i is an isomorphism $I_{Z_i}|_\ell$ with i-th factor of $\mathcal{O}_\ell^2$. e.g. $n = 7$, $I_{Z_1} = (x^2, xy, y^2)$ and $I_{Z_2} = (x^2, y^2)$.

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- There is a bijection between $M(n)^{\widetilde{T}}$ and the set of pairs of *Young diagrams* $\vec{Y} = (Y_1, Y_2)$ such that that $|Y_1| + |Y_2| = n$.

- Notation. For $\alpha, \beta \in \{0, 1\}$ denote the $\widetilde{T}$-character (resp. equivariant Euler class) of $\mathrm{Ext}^1(I_{Z_\alpha}, I_{Z_\beta}(-\ell))$ by $N^{\vec{Y}}_{\alpha,\beta}(t_1, t_2, e)$ (resp. $n^{\vec{Y}}_{\alpha,\beta}(s_1, s_2, a)$). Here, $(t_1, t_2, e) = (e^{s_1}, e^{s_2}, e^a)$.
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- As a power series in Λ, Z^{inst} starts with 1. Define

$$F^{\mathrm{inst}}(s_1, s_2, a, \Lambda, \vec{\tau}) := \log Z^{\mathrm{inst}}(s_1, s_2, a, \Lambda, \vec{\tau}).$$

- Notation. Define c_n by $\frac{1}{(e^{s_1 t}-1)(e^{s_2 t}-1)} = \sum_{n\geqslant 0} \frac{c_n}{n!} t^{n-2}$ and

$$\gamma_{s_1,s_2}(x,\Lambda) := \frac{1}{s_1 s_2}\Big(-\frac{1}{2}x^2\log(x/\Lambda)+\frac{3}{4}x^2\Big)+\frac{s_1+s_2}{2s_1 s_2}\Big(-x\log(x/\Lambda)+x\Big) - \frac{s_1^2+s_2^2+3s_1 s_2}{12 s_1 s_2}\log(x/\Lambda)+\sum_{n=3}^{\infty}\frac{c_n x^{2-n}}{n(n-1)(n-2)}.$$

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- Define $F(s_1,s_2,a,\Lambda,\vec{\tau}) := F^{\text{pert}}(s_1,s_2,a,\Lambda)+F^{\text{inst}}(s_1,s_2,a,\Lambda,\vec{\tau})$.

- Define a family of elliptic curves C_u: $y^2 = (z^2 - u)^2 - 4\Lambda^4$ parameterized by $u \in \mathbb{C}$, called the u-plane. The parameter Λ is called the *renormalization scale*. When $\Lambda = 0$ the theory goes to the *classical limit*. C_u is singular for $u = \pm 2\Lambda^2$.

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- The *SW prepotential* $\mathcal{F}_0$ is a locally defined function on the u-plane satisfying $a^D = -\frac{\partial \mathcal{F}_0}{\partial a}$. After a suitable choice of branch of log it can be viewed as a holomorphic function of a, Λ on some domain.

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- Nekrasov conjecture: $s_1 s_2 F(s_1, s_2, a, \Lambda)$ is regular at $s_1 = 0 = s_2$ and moreover

$$s_1 s_2 F(s_1, s_2, a, \Lambda)|_{s_1=0=s_2} = \mathcal{F}_0(a, \Lambda).$$

  This is known to be a natural relation from a physical point of view and is similar to the mirror symmetry in which Nakrasov's partition function is a counterpart of GW invariants on the symplectic side and the SW prepotential is on the complex side. This conjecture is proven by **Nakajima-Yoshioka**, **Nekrasov-Okounkov**, and **Braverman-Etingof** by different methods.
- The idea of the first proof is to consider a similar partition functions (with insertions) via the framed moduli spaces of rank 2 torsion free sheaves on the blow up of $\mathbb{P}^2$, and prove a blow up formula relating it to Nekarsov's partition function. For some choices of insertions the partition functions of the blowup are shown to vanish. These give a differential equation satisfied by Nekrasov's partition function, which turns out to essentially be the same differential equation satisfied by SW prepotential.

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- The cobordism class of a stably complex manifold is completely determined by the collection of its Chern numbers. As a corollary of the above reuslt, the class of $X^{[n]}$ in Ω, the complex cobordism ring with rational coefficients, depends only on the class $[X] \in \Omega_2$.

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- Application: The formula

$$\chi_{-y}\Big(\sum_{n=0}^{\infty}[X^{[n]}]z^n\Big) = \exp\Big(\sum_{m=1}^{\infty}\frac{\chi_{-y^m}(X)}{1-(yz)^m}\frac{z^m}{m}\Big)$$

  (and similarly for the Poincaré polynomials) can be proven by noting that both sides are multiplicative in $[X]$ and hence reducing the proof to the cases $X = \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, and then applying toric techniques.

- Notation. Let $b_1, \ldots, b_s$ be a homogeneous basis of $H_*(X)$. For $\rho \geqslant 1$, let $\tau_1^\rho, \ldots, \tau_s^\rho$ be indeterminates, put $\alpha_\rho := \sum_{k=1}^s q_k^\rho b_k \tau_k^\rho$ with $q_k^\rho \in \mathbb{Q}$ and define a generating series for Donaldson invariants

$$D_{c_1}^H(\exp(\sum_{\rho\geqslant 1} \alpha_\rho)) := \sum_{d\geqslant 0} \Lambda^d \int_{M^H(c_1,d)} \exp(\sum_{\rho\geqslant 1} \mu_\rho(\alpha_\rho)),$$

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- Wall-crossing terms.

$$\delta_{\xi,t}(\exp(\sum_{\rho \geqslant 1} \alpha_\rho)) := \sum_{l \geqslant 0} \Lambda^{4l-\xi^2-3\chi(\mathcal{O}_X)} \cdot \int_{X_2^{[l]}} \frac{\exp\left(\sum_{\rho\geqslant 1}(-1)^\rho \left[\operatorname{ch}(\mathcal{I}_1)e^{\frac{\xi-t}{2}} + ch(\mathcal{I}_2)e^{\frac{t-\xi}{2}}\right]_{\rho+1}/\alpha_\rho\right)}{c^{-t}(\mathcal{A}_{\xi,-})c^t(\mathcal{A}_{\xi,+})} \in \Lambda^{-\xi^2-3\chi(\mathcal{O}_X)}\mathbb{Q}[t,t^{-1}][[\Lambda,(\tau_k^\rho)]],$$

where $\mathcal{I}_i$ are the pullbacks of the universal ideal sheaves to $X \times X_2^{[n]}$, and $\dfrac{1}{c^t(E)} := \dfrac{1}{t^r}\dfrac{1}{\sum_{i=1}^r c_i(E)\frac{1}{t^i}} H^*(-)[[t^{-1}]]$ for any rank r vector bundle E.

- Taking the coefficient of t^{-1}

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$$D_{c_1}^{H_+}(\exp(\sum_{\rho\geqslant 1}\alpha_\rho)) - D_{c_1}^{H_-}(\exp(\sum_{\rho\geqslant 1}\alpha_\rho)) = \sum_{\xi\in B_+}\delta_\xi(\exp(\sum_{\rho\geqslant 1}\alpha_\rho)).$$

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- Remark. This formula is shown to be compatible with **Fintushel-Stern**'s blowup formula and so it suffices to be proven after blowing up X at sufficiently many points. Hence one may assume $M^{H_\pm}(c_1, d)$ is of expected dimension without loss of generality. The key idea of the proof is that passing the wall W^ξ the moduli space changes by replacing certain sheaves lying in extensions of ideal sheaves of zero-dimensional schemes twisted by line bundles by extensions the other way round:

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- **Mochizuki** proves the same result for general walls using virtual fundamental classes and virtual localization. When ξ is not good $\mathcal{A}_{\xi,\pm}$ are not necessarily vector bundles and are replaced by the corresponding classes in K-theory.

- Suppose that Y is a smooth projective toric surface e.g. $Y = \mathbb{P}^2$. This means that Y contains $\Gamma = \mathbb{C}^{*2}$ as an open subset and the action of Γ extends to Y. There are finitely many fixed points $p_1, \ldots, p_\chi$, where χ is the Euler number of Y. Let $w(x_i), w(y_i)$ be the weights of the Γ-action on T_{Y,p_i}.

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- One may define *equivariant Donaldson invariants* of Y for the equivariant lifts of (co)homology classes by means of the moduli space of equivariant semistable sheaves. Denote the generating series and wall-crossing terms by $\widetilde{D}^H_{c_1}(-)$ and $\widetilde{\delta}_{\xi,t}(-)$, respectively. They specialize to $D^H_{c_1}(-)$ and $\delta_{\xi,t}(-)$ by setting $s_1 = 0 = s_2$.

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- **Theorem 2 (Göttsche-Yoshioka-Nakajima)** For Y toric

$$\widetilde{\delta}_{\xi,t}(\exp(\sum_{\rho\geqslant 1}\alpha_\rho)) =$$

$$\frac{1}{\Lambda}\exp\Big(\sum_{i=1}^{\chi} F(w(x_i), w(y_i), \frac{t-\xi|_{p_i}}{2}, \Lambda, ((-1)^\rho \alpha_\rho|_{p_i})_\rho)\Big)$$

as elements of the ring $\Lambda^{-\xi^2-3}\mathbb{Q}[s_1, s_2]((t^{-1}))[[\Lambda, (\tau_k^\rho)]]$.

- For $\tau \in \mathbb{H}$ let $q := e^{2\pi i\tau}$ and define the theta functions

$$\theta_{00}(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2/2}, \ \theta_{01}(\tau) := \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2}, \ \theta_{10}(\tau) := \sum_{n\in\mathbb{Z}} q^{(n+1)^2/8}.$$

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- *Normalized Eisenstein series of weight 2*: $E_2(\tau) := 1 - 24\sum_{n\in\mathbb{Z}} \sigma_1(n)q^n$.
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- Going back to the u-plane, the period of C_u is given by $\tau = \frac{-1}{2\pi i}\frac{\partial^2 \mathcal{F}_0}{(\partial a)^2}$ and $q = \exp(-\frac{\partial^2 \mathcal{F}_0}{(\partial a)^2})$. Then it can be shown

$$u = \frac{\theta_{00}^4 + \theta_{10}^4}{\theta_{00}^2\theta_{10}^2}\Lambda^2, \quad \frac{du}{da} = \frac{2i}{\theta_{00}\theta_{10}}\Lambda, \quad a = \frac{2E_2 + \theta_{00}^4 + \theta_{10}^4}{3\theta_{00}\theta_{10}}\Lambda.$$

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- **Theorem 3 (Göttsche-Yoshioka-Nakajima)** Let ξ be a good class and Y be toric.

$$\delta_\xi(\exp(\alpha z + px)) = i^{\xi\cdot K_Y - 1}\big[\Delta\big]_{q^0},$$

where $\alpha \in H_2(X,\mathbb{Z})$ and $p \in H_0(X,\mathbb{Z})$ is the class of a point and

$$\Delta := q^{-\xi^2/8}\exp\big(\frac{du}{da}\langle\alpha,\xi/2\rangle z + T\alpha^2 z^2 - ux\big)\big(\frac{i}{\Lambda}\frac{du}{da}\big)^3 \theta_{01}^{K_Y^2}.$$

This result is proven by the localization formula using Theorem 2 and Nekrasov's conjecture.

- **Theorem 4 (Göttsche-Yoshioka-Nakajima)** There exists universal power series $A_1, \ldots, A_8 \in \mathbb{Q}((t^{-1}))[[\Lambda]]$ such that for all smooth projective surfaces X and $\xi \in \mathrm{Pic}(X)$

$$(-1)^{\chi(\mathcal{O}_X)+\xi(\xi-K_X)/2} t^{-\xi^2-2\chi(\mathcal{O}_X)} \Lambda^{\xi^2+3\chi(\mathcal{O}_X)} \delta_{\xi,t}(\exp(\alpha z + px))$$
$$= \exp\big(\xi^2 A_1 + \xi \cdot c_1(X) A_2 + c_1(X)^2 A_3 + c_2(X) A_4 + \alpha \cdot \xi A_5 z$$
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- Sketch of proof of Theorem 5: Substituting $t = 2a$, we can rewrite Theorem 4 in terms of q. For any triple (X, ξ, β) there exists $\vec{v}(X, \xi, \beta) = (v_1, \dots, v_8) \in \mathbb{Q}^8$ such that

$$\delta_{\xi,t}(\beta) = \frac{i^{\xi\cdot K_X} q^{-\xi^2/8}}{\Lambda^{\chi(\mathcal{O}_X)}} \Big(\frac{i}{\Lambda}\frac{du}{da}\Big)^{2\chi(\mathcal{O}_X)}\Big) \exp\Big(\sum_{i=1}^{8} v_i B_i\Big)$$

for some universal power series $B_i \in \mathbb{C}((q^{-1/8}))[[\Lambda]]$.
$\vec{v}(X, \xi, \beta)$ with X a toric surface and ξ a good class generate $\mathbb{Q}^8$ as a vector space, and so B_i are determined by their values for toric surfaces and good classes.